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Mechanics of Trajectory Optimization Using Nonsingular Variational Equations in Polar Coordinates

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The complete set of the state and adjoint differential equations for the nonsingular equinoctial orbit elements expressed in polar coordinates and written in terms of the true longitude is presented. Previous formulations adopted the equinoctial frame as the orbital frame for component resolution of the various perturbation accelerations. The consideration of the position dependency of the J_2 acceleration in the context of precision integrated orbital transfer trajectories led us to adopt the more convenient polar frame coupled with the use of the true longitude as the accessory variable needed in the description of the variational equations, inasmuch as it is more difficult to generate the contribution of the J_2 perturbation to the adjoint equations if the variational equations are left in terms of the eccentric longitude.

I. Introduction

THE consideration of the nonsingular equinoctial orbit elements has been of great benefit in trajectory propagation and optimization. References 1–7 made use of the corresponding variational equations resolved in the equinoctial orbital coordinate system. In Refs. 8–12, modern numerical methods based on the techniques of collocation and nonlinear programming are applied to the solution of low-thrust Earth-bound and interplanetary trajectories. These direct methods have introduced robustness characteristics to the optimization process and were first used with great success by launch vehicle trajectory optimization and performance analysts. In particular, Betts[9,10] uses a variety of equinoctial element sets within the framework of the direct method. Following the more traditional approach based on the calculus of variations and the indirect shooting method, this paper develops the differential equations for the adjoint or multiplier variables to be numerically integrated simultaneously with the state differential equations to solve the two-point-boundary-value orbit transfer problem. As noted by Betts, the elements of the matrix of partial derivatives of the Hamiltonian with respect to the equinoctial elements derived here in analytic form can be used to develop a certain analytic Jacobian matrix in connection with nonlinear programming (NLP) solvers used with direct methods, thereby saving considerable computation time with respect to the adoption of a numerically determined Jacobian. The variational equations are usually expressed in terms of the eccentric longitude, which can either be left as an accessory variable or be adopted as the fast or sixth orbital element.

In Ref. 3, the averaged J_2 perturbation effect was accounted for in the design of the optimal transfer trajectory. However, when the exact or rather position-dependent J_2 effect must be considered in the context of precision integrated transfer solutions, the J_2 perturbation acceleration components become dependent on the eccentric longitude. Because these components are given in the rotating Euler–Hill or polar frame, it is much simpler to use this frame as the orbital frame instead of the equinoctial frame because otherwise the transformed expressions for these components become more complicated. These expressions are still quite complicated if expressed in terms of the eccentric longitude even if the polar frame is used. This is important because we must take the partial derivatives of these expressions with respect to the elements to generate the adjoint differential equations. This task is made easier by expressing the variational equations for the J_2 perturbation in terms of the true longitude, and to be completely consistent, the equations for the thrust perturbation will also be expressed in polar coordinates and in terms of the true longitude. This paper shows the mathematical derivations that lead to the generation of the full set of the dynamic and adjoint differential equations in polar coordinates in terms of the true longitude with the mean longitude as the sixth element. The dynamic equations, which are also found in Ref. 13, are directly obtained from the equations for the classical elements in the Gaussian form. The J_2 perturbation equations will appear in a later effort, as well as a slightly simpler set, which uses the true longitude as the sixth element itself. We duplicate an example of an optimal transfer generated earlier with a previously validated formulation in order to validate the mathematics of this new version.

II. Dynamic and Adjoint Differential Equations in Polar Coordinates

The variational equations for the classical elements in the Gaussian form with the components of the disturbing acceleration in the radial, transverse, and out-of-plane directions are given by

$$\dot{a} = (2a^2/j)[es_{\theta^*} f_r + (b/r) f_\theta] \tag{1}$$

$$\dot{e} = (1/j)\{bs_{\theta^*} f_r + [(b+r)c_{\theta^*} + re] f_\theta\} \tag{2}$$

$$\dot{i} = (rc_\theta/j) f_h \tag{3}$$

$$\dot{\Omega} = (rs_\theta/js_i) f_h \tag{4}$$

$$\dot{\omega} = \frac{1}{je}[-bc_{\theta^*} f_r + (b+r)s_{\theta^*} f_\theta] - \frac{rs_\theta c_i}{js_i} f_h \tag{5}$$

$$\dot{M} = n + \frac{(1-e^2)^{\frac{1}{2}}}{je}[(bc_{\theta^*} - 2re) f_r - (b+r)s_{\theta^*} f_\theta] \tag{6}$$

where $n = \mu^{1/2} a^{-3/2}$ is the orbit mean motion, with $\mu = 398{,}601.3$ km^3/s^2, the Earth gravity constant; a, e, i, Ω, ω, and M are the classical orbit elements; $b = a(1-e^2)$ is the orbit parameter; $j = [\mu a(1-e^2)]^{1/2}$ is the orbit angular momentum; r is the radial distance defined by $r = b/(1 + ec_{\theta^*})$; θ is the angular position defined by $\theta = \omega + \theta^*$, with θ^* the true anomaly; and f_r, f_θ, and f_h are the components of the disturbing acceleration vector resolved along the rotating coordinate axes $\hat{r}$, $\hat{\theta}$, and $\hat{h}$, whose directions have been defined earlier. This set of equations, which is singular at zero eccentricity and zero inclination, can be converted to a nonsingular set involving the equinoctial elements defined as $a = a$, $h = e\sin(\omega + \Omega)$, $k = e\cos(\omega + \Omega)$, $p = \tan(i/2)\sin\Omega$, $q = \tan(i/2)\cos\Omega$, and $\lambda = M + \omega + \Omega$, with λ the mean longitude. Let $\bar{\omega} = \omega + \Omega$ be the longitude of pericenter, with $F = E + \bar{\omega}$ and $L = \theta^* + \bar{\omega}$ the eccentric and true longitudes, respectively, E being

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the eccentric anomaly. We have s_{θ^*} and c_{θ^*} standing for $\sin\theta^*$ and $\cos\theta^*$, etc.,

$$s_{\theta^*} = s_L c_{\bar\omega} - s_{\bar\omega} c_L \qquad c_{\theta^*} = c_L c_{\bar\omega} + s_L s_{\bar\omega}$$

and because L can also be written as $L = \Omega + \theta$,

$$s_\theta = s_L c_\Omega - c_L s_\Omega \qquad c_\theta = c_L c_\Omega + s_L s_\Omega$$

In Eqs. (1–5), we have used j and b for the angular momentum and orbit parameter, respectively, to avoid confusion with the symbols used for the equinoctial elements h and p. The radial distance r can be written as

$$r = a(1 - kc_F - hs_F) \tag{7}$$

or in terms of the true longitude as

$$r = \frac{a(1 - h^2 - k^2)}{1 + hs_L + kc_L} \tag{8}$$

We can also write

$$j = na^2(1 - h^2 - k^2)^{\frac{1}{2}} \qquad \frac{r}{j} = \frac{(1 - h^2 - k^2)^{\frac{1}{2}}}{na(1 + hs_L + kc_L)}$$

$$1 + \frac{b}{r} = 2 + hs_L + kc_L$$

and with

$$\beta = \frac{1}{1 + (1 - h^2 - k^2)^{\frac{1}{2}}} \tag{9}$$

we have

$$(1 - h^2 - k^2)^{\frac{1}{2}} = [(1 - \beta)/\beta] \tag{10}$$

$$h^2 + k^2 = [(2\beta - 1)/\beta^2] \tag{11}$$

Also, $\partial\beta/\partial h = h\beta^3/(1-\beta)$ and $\partial\beta/\partial k = k\beta^3/(1-\beta)$. Finally, Kepler's equation is written in terms of the eccentric longitude F as

$$\lambda = F - ks_F + hc_F \tag{12}$$

Using the identities $c_i = c_{i/2}^2 - s_{i/2}^2$ and $s_i = 2s_{i/2}c_{i/2}$, the equations of motion (1–6) can now be transformed to the following nonsingular set:

$$\dot a = \frac{2}{n(1 - h^2 - k^2)^{\frac{1}{2}}}[(ks_L - hc_L)f_r + (1 + hs_L + kc_L)f_\theta] \tag{13}$$

$$\dot h = \frac{(1 - h^2 - k^2)^{\frac{1}{2}}}{na(1 + hs_L + kc_L)}\{-(1 + hs_L + kc_L)c_L f_r + [h + (2 + hs_L + kc_L)s_L]f_\theta - k(pc_L - qs_L)f_h\} \tag{14}$$

$$\dot k = \frac{(1 - h^2 - k^2)^{\frac{1}{2}}}{na(1 + hs_L + kc_L)}\{(1 + hs_L + kc_L)s_L f_r + [k + (2 + hs_L + kc_L)c_L]f_\theta + h(pc_L - qs_L)f_h\} \tag{15}$$

$$\dot p = \frac{(1 - h^2 - k^2)^{\frac{1}{2}}}{2na(1 + hs_L + kc_L)}(1 + p^2 + q^2)s_L f_h \tag{16}$$

$$\dot q = \frac{(1 - h^2 - k^2)^{\frac{1}{2}}}{2na(1 + hs_L + kc_L)}(1 + p^2 + q^2)c_L f_h \tag{17}$$

$$\dot\lambda = n - \frac{(1 - h^2 - k^2)^{\frac{1}{2}}}{na(1 + hs_L + kc_L)}\Big\{\Big[\beta(1 + hs_L + kc_L)(hs_L + kc_L) + 2(1 - h^2 - k^2)^{\frac{1}{2}}\Big]f_r + \beta(2 + hs_L + kc_L) \times (hc_L - ks_L)f_\theta + (pc_L - qs_L)f_h\Big\} \tag{18}$$

These equations are essentially identical to those found in Ref. 13. It is clear that as λ is being integrated, the eccentric longitude must first be solved from Kepler's equation, after which the true longitude is obtained from the following expressions:

$$c_L = (a/r)\Big[(1 - h^2\beta)c_F + hk\beta s_F - k\Big] \tag{19}$$

$$s_L = (a/r)\Big[hk\beta c_F + (1 - k^2\beta)s_F - h\Big] \tag{20}$$

These equations are obtained from the components of the radius vector along the equinoctial directions $\hat f$ and $\hat g$ such that

$$X_1 = rc_L = a\Big[(1 - h^2\beta)c_F + hk\beta s_F - k\Big] \tag{21}$$

$$Y_1 = rs_L = a\Big[hk\beta c_F + (1 - k^2\beta)s_F - h\Big] \tag{22}$$

The $(\hat f, \hat g, \hat w)$ equinoctial coordinate frame is defined in terms of the classical orbit elements inasmuch as $\hat f$ and $\hat g$ are contained in the instantaneous orbit plane with $\hat f$ obtained through a clockwise rotation of an angle Ω from the direction of the ascending node. Furthermore, h and k are the components of the eccentricity vector $\boldsymbol{e}$ along $\hat f$ and $\hat g$, with the elements p and q defining the matrix that rotates the orbit frame $(\hat f, \hat g, \hat w)$ into the inertial frame $(\hat x, \hat y, \hat z)$. Equations (13–18) are written in polar coordinates $\hat r$, $\hat\theta$, $\hat h$ instead of the $\hat f$, $\hat g$, $\hat w$ equinoctial coordinates. The transformation between these two orbital frames is simply given by

$$\begin{pmatrix} \hat r \\ \hat\theta \\ \hat h \end{pmatrix} = \begin{pmatrix} X_1/r & Y_1/r & 0 \\ -Y_1/r & X_1/r & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat f \\ \hat g \\ \hat w \end{pmatrix} \tag{23}$$

The equinoctial frame is needed here only to define the true longitude L through Eqs. (19) and (20) because $L = f(h, k, F)$ with $F = f(h, k, \lambda)$. Once we derive the differential equations for the adjoints, we will not worry about the equinoctial frame because the optimal thrust direction will be directly computed in the polar coordinate frame with the thrust pitch and yaw angles θ_t and θ_h obtained directly from

$$\theta_t = \tan^{-1}(u_r/u_\theta) \tag{24}$$

$$\theta_h = \tan^{-1}(u_w/u_\theta) \tag{25}$$

with $\hat u = (u_r, u_\theta, u_w)$ such that the thrust is given by $\boldsymbol{f} = f\hat u$, with f its magnitude and $\hat u$ its direction, and the acceleration is given by $\boldsymbol{\Gamma} = \boldsymbol{f}/m = f_t\hat u = (f_r, f_\theta, f_h)$, with m the vehicle mass and $f_t = f/m$. Considering continuous constant acceleration for the purpose of the discussion and without any loss of generality, let $\boldsymbol{z} = (a\ h\ k\ p\ q\ \lambda)^T$ be the state vector and let $\boldsymbol{\lambda}_z = (\lambda_a\ \lambda_h\ \lambda_k\ \lambda_p\ \lambda_q\ \lambda_\lambda)^T$ be the corresponding adjoint vector. Note that when λ appears with one of the elements as a subscript, it is the multiplier adjoint to that particular element, whereas without the susbscript, it is the mean longitude. Let the 6×3 B matrix be such that

$$\begin{pmatrix} \dot a \\ \dot h \\ \dot k \\ \dot p \\ \dot q \\ \dot\lambda \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \\ B_{41} & B_{42} & B_{43} \\ B_{51} & B_{52} & B_{53} \\ B_{61} & B_{62} & B_{63} \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \\ u_h \end{pmatrix} f_t + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ n \end{pmatrix} \tag{26}$$

The six rows of B consist of the partials of the six equinoctial elements with respect to the velocity vector, with the components along the polar $\hat r$, $\hat\theta$, $\hat h$ axes. The Hamiltonian and the Euler–Lagrange equations can now be written as

$$H = \boldsymbol{\lambda}_z^T B(z, L) f_t \hat u + \lambda_\lambda n = \boldsymbol{\lambda}_z^T \dot{\boldsymbol{z}} \tag{27}$$

$$\dot{\boldsymbol{\lambda}}_z = -\frac{\partial H}{\partial \boldsymbol{z}} = -\boldsymbol{\lambda}_z^T \frac{\partial B}{\partial \boldsymbol{z}} f_t \hat u - \lambda_\lambda \frac{\partial n}{\partial \boldsymbol{z}} \tag{28}$$

When taking the partial derivatives in Eq. (28), we must allow for the variation of L with respect to h, k, and λ, and because L is given in terms of F, we must also allow for the variation of F with respect to h, k, and λ:

$$\frac{\partial L}{\partial h}=\left(\frac{\partial L}{\partial h}\right)_F+\frac{\partial L}{\partial F}\frac{\partial F}{\partial h} \tag{29}$$

$$\frac{\partial L}{\partial k}=\left(\frac{\partial L}{\partial k}\right)_F+\frac{\partial L}{\partial F}\frac{\partial F}{\partial k} \tag{30}$$

$$\frac{\partial L}{\partial \lambda}=\frac{\partial L}{\partial F}\frac{\partial F}{\partial \lambda} \tag{31}$$

with the subscript F indicating that F is held constant there. We also need $\partial r/\partial a$, $\partial r/\partial h$, $\partial r/\partial k$, and $\partial r/\partial\lambda$, and it will be shown that $\partial L/\partial a=0$. These partial derivatives must be expressed in terms of L such that both the dynamic and adjoint equations are written consistently in terms of L. Because we are integrating λ, it effectively becomes independent of a, h, and k such that, from Kepler's equation $\lambda=F-ks_F+hc_F$, we have from $\partial\lambda/\partial a=(r/a)(\partial F/\partial a)=0$ that $\partial F/\partial a=0$. In a similar way, from $\partial\lambda/\partial h=0$ we obtain $\partial F/\partial h=-(a/r)c_F$, from $\partial\lambda/\partial k=0$ we have $\partial F/\partial k=(a/r)s_F$, and finally from $\partial\lambda/\partial\lambda=1$ we have $\partial F/\partial\lambda=a/r$. This in turn yields the partial derivatives of the radial distance r with respect to a, h, k, and λ such that from $r=a(1-kc_F-hs_F)$ we get

$$\frac{\partial r}{\partial a}=\frac{r}{a} \tag{32}$$

$$\frac{\partial r}{\partial h}=\frac{a^2}{r}(h-s_F) \tag{33}$$

$$\frac{\partial r}{\partial k}=\frac{a^2}{r}(k-c_F) \tag{34}$$

$$\frac{\partial r}{\partial F}=a(ks_F-hc_F) \tag{35}$$

All of these partials are identical to those derived in Ref. 6, which used the elements a, h, k, p, q, and λ and the equinoctial orbit frame $\hat{f}$, $\hat{g}$, $\hat{w}$ and with all of the differential equations written in terms of the eccentric longitude F. In particular, if we substitute the quantities rc_L and rs_L for X_1 and Y_1, then the expressions for s_F and c_F in Refs. 1–3 and 5 can be written as

$$s_F=h+\frac{r}{a}\frac{s_L(1-h^2\beta)-hk\beta c_L}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{36}$$

$$c_F=k+\frac{r}{a}\frac{c_L(1-k^2\beta)-hk\beta s_L}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{37}$$

yielding the following identity:

$$ks_F-hc_F=\frac{r}{a}\frac{ks_L-hc_L}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{38}$$

These expressions now allow us to write $\partial r/\partial h$, $\partial r/\partial k$, and $\partial r/\partial F$ directly in terms of L, namely,

$$\frac{\partial r}{\partial h}=-\frac{a\left[(1-h^2\beta)s_L-hk\beta c_L\right]}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{39}$$

$$\frac{\partial r}{\partial k}=-\frac{a\left[(1-k^2\beta)c_L-hk\beta s_L\right]}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{40}$$

$$\frac{\partial r}{\partial F}=\frac{a(ks_L-hc_L)}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{41}$$

Now the partial derivative $\partial L/\partial F$ needed in Eqs. (29–31) can be obtained from Eq. (19) as follows. From $c_L=X_1/r$,

$$-s_L\frac{\partial L}{\partial F}=\frac{1}{r}\frac{\partial X_1}{\partial F}-\frac{X_1}{r^2}\frac{\partial r}{\partial F}$$

with $\partial r/\partial F$ given in Eq. (41) and

$$\frac{\partial X_1}{\partial F}=a\left[hk\beta c_F-(1-h^2\beta)s_F\right]$$

Using Eqs. (36) and (37) in $\partial X_1/\partial F$, we get

$$\frac{\partial L}{\partial F}=\frac{c_L(ks_L-hc_L-hk)}{s_L(1-h^2-k^2)^{\frac{1}{2}}}+\frac{ah}{rs_L}(1-h^2-k^2)^{\frac{1}{2}}+\frac{1-h^2}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{42}$$

In a similar way, from Eq. (20) and using

$$\frac{\partial Y_1}{\partial F}=a\left[(1-k^2\beta)c_F-hk\beta s_F\right]$$

we have

$$\frac{\partial L}{\partial F}=\frac{ak}{rc_L}(1-h^2-k^2)^{\frac{1}{2}}+\frac{1-k^2}{(1-h^2-k^2)^{\frac{1}{2}}}-\frac{s_L(ks_L-hc_L+hk)}{c_L(1-h^2-k^2)^{\frac{1}{2}}} \tag{43}$$

The partials in Eqs. (42) and (43) can be shown to be identical, and they can further be reduced to the simpler form

$$\frac{\partial L}{\partial F}=\frac{a}{r}(1-h^2-k^2)^{\frac{1}{2}} \tag{44}$$

or

$$\frac{\partial L}{\partial F}=\frac{1+hs_L+kc_L}{(1-h^2-k^2)^{\frac{1}{2}}} \tag{45}$$

We can also obtain $\partial L/\partial F$ more simply from

$$\tan L=\frac{Y_1}{X_1}\qquad \frac{1}{c_L^2}\frac{\partial L}{\partial F}=\frac{X_1(\partial Y_1/\partial F)-Y_1(\partial X_1/\partial F)}{X_1^2}$$

and leaving X_1, Y_1, $\partial X_1/\partial F$, and $\partial Y_1/\partial F$ in terms of F while using $X_1^2=r^2c_L^2$ to get

$$\frac{r^2}{a^2}\frac{\partial L}{\partial F}=(1-h^2\beta-k^2\beta)(1-hs_F-kc_F)$$

We do not need to replace s_F and c_F by their expressions in terms of s_L and c_L because the last parenthesis is conveniently replaced by r/a such that, finally and more readily than before,

$$\begin{aligned}\frac{\partial L}{\partial F}&=\frac{a}{r}(1-h^2\beta-k^2\beta)\\&=\frac{a}{r}\frac{1-\beta}{\beta}=\frac{a}{r}(1-h^2-k^2)^{\frac{1}{2}}\end{aligned}$$

Now as far as the $\partial L/\partial a$ partial is concerned, we can use c_L in Eq. (19) to get

$$-s_L\frac{\partial L}{\partial a}=\left[\left(r-a\frac{\partial r}{\partial a}\right)\Big/r^2\right]\left(\frac{r}{a}c_L\right)+\frac{a}{r}\left[hk\beta c_F-(1-h^2\beta)s_F\right]\frac{\partial F}{\partial a} \tag{46}$$

with

$$\frac{\partial r}{\partial a}=\frac{r}{a}+a(ks_F-hc_F)\frac{\partial F}{\partial a} \tag{47}$$

However, $\partial F/\partial a=0$ and, therefore, $\partial r/\partial a=r/a$, and because s_L is not always equal to zero, we have

$$\frac{\partial L}{\partial a}=0 \tag{48}$$

If we delay taking $\partial F/\partial a = 0$ in Eqs. (46) and (47) and make use of Eq. (36) for s_F and Eq. (37) for c_F as well as the identity in Eq. (38), then the expression in Eq. (46) will be reduced to

$$\frac{\partial L}{\partial a} = \frac{1 + hs_L + kc_L}{(1 - h^2 - k^2)^{\frac{1}{2}}}\frac{\partial F}{\partial a} = \frac{\partial L}{\partial F}\frac{\partial F}{\partial a} \tag{49}$$

where the partial in Eq. (45) has been recognized. Now because $\partial F/\partial a = 0$, then $\partial L/\partial a$ must also be identically equal to zero because $\partial L/\partial F$ itself is always nonzero. There are several ways of generating the $\partial L/\partial h$ and $\partial L/\partial k$ partials. We can use Eq. (19) or (20) and write, for example,

$$\frac{\partial L}{\partial h} = -\frac{1}{s_L}\frac{\partial}{\partial h}\left\{\frac{a}{r}\Big[(1 - h^2\beta)c_F + hk\beta s_F - k\Big]\right\}$$

and allow for the variation of F with respect to h by using $\partial F/\partial h = -(a/r)c_F$, as well as using Eq. (39) for $\partial r/\partial h$. Then s_F and c_F are replaced by their expressions in terms of L according to Eqs. (36) and (37) such that the preceding partial is given strictly in terms of L. We can also use $\tan L = Y_1(F)/X_1(F)$ and write

$$\frac{\partial L}{\partial h} = \frac{X_1(\partial Y_1/\partial h) - Y_1(\partial X_1/\partial h)}{r^2}$$

and express the right-hand side in terms of L. It is, however, much easier to use $r = a(1 - h^2 - k^2)\cdot(1 + hs_L + kc_L)^{-1}$ and write

$$\frac{\partial r}{\partial h} = \frac{-2ah}{(1 + hs_L + kc_L)} - \frac{a(1 - h^2 - k^2)}{(1 + hs_L + kc_L)^2} \times \left[s_L + (hc_L - ks_L)\frac{\partial L}{\partial h}\right]$$

and make use of the $\partial r/\partial h$ partial in Eq. (39) written directly in terms of L. Equating these two partials will yield the desired quantity

$$\frac{\partial L}{\partial h} = \frac{(a^2/r^2)(1 - h^2 - k^2)^{\frac{1}{2}}\Big[(1 - h^2\beta)s_L - hk\beta c_L\Big] - s_L - (2ah/r)}{hc_L - ks_L} \tag{50}$$

Let us derive this particular partial in still another way. From $L = \omega + \Omega + \theta^* = \tan^{-1}(h/k) + \theta^*$, we have, with $e^2 = h^2 + k^2$,

$$\frac{\partial L}{\partial h} = \frac{k}{e^2} + \frac{\partial \theta^*}{\partial h} \tag{51}$$

Now from $r = a(1 - e^2)/(1 + ec_{\theta^*})$, we have also

$$\frac{\partial r}{\partial h} = \left[\left(-2ae\frac{\partial e}{\partial h}\right)\Big/(1 + ec_{\theta^*})\right] - \frac{a(1 - e^2)}{(1 + ec_{\theta^*})^2} \times \left[\frac{\partial e}{\partial h}c_{\theta^*} - es_{\theta^*}\frac{\partial \theta^*}{\partial h}\right] \tag{52}$$

However, $\partial e/\partial h = h/e$, and from orbital mechanics we have $rc_{\theta^*} = a(c_E - e)$ and $rs_{\theta^*} = a(1 - e^2)^{1/2}s_E$, where E, the eccentric anomaly, is given by $E = F - (\omega + \Omega)$. Therefore,

$$c_E = (k/e)c_F + (h/e)s_F \qquad s_E = (k/e)s_F - (h/e)c_F$$

such that with $e = (h^2 + k^2)^{1/2}$,

$$c_{\theta^*} = (a/er)\Big[kc_F + hs_F - e^2\Big]$$

$$s_{\theta^*} = (a/er)(1 - e^2)^{\frac{1}{2}}(ks_F - hc_F)$$

Once again using $\partial r/\partial h$ of Eq. (39), we can solve for $\partial\theta^*/\partial h$ from Eq. (52) so that finally Eq. (51) is given directly in terms of L:

$$\frac{\partial L}{\partial h} = \frac{k}{h^2 + k^2} - \frac{a^2}{r^2}\frac{(1 - h^2 - k^2)^{\frac{1}{2}}}{ks_L - hc_L}\Big[(1 - h^2\beta)s_L - hk\beta c_L\Big] + \frac{a}{r}\frac{h}{ks_L - hc_L}\left[2 + \frac{1 - h^2 - k^2}{h^2 + k^2}\right] - \frac{h}{(h^2 + k^2)(ks_L - hc_L)}$$

After further regrouping of terms, it can be shown that this partial is identical to Eq. (50). This partial can also be given in terms of F as

$$\frac{\partial L}{\partial h} = \frac{(-a^2/r^2)(s_F - h)(1 - h^2 - k^2) + Y_1/a + 2h}{(1 - h^2 - k^2)^{\frac{1}{2}}(ks_F - hc_F)} \tag{53}$$

In a similar manner, we can derive $\partial L/\partial k$ and get

$$\frac{\partial L}{\partial k} = \frac{(a^2/r^2)(1 - h^2 - k^2)^{\frac{1}{2}}\Big[(1 - k^2\beta)c_L - hk\beta s_L\Big] - c_L - 2(a/r)k}{hc_L - ks_L} \tag{54}$$

in terms of L, or as in the next equation, in terms of F:

$$\frac{\partial L}{\partial k} = \frac{(-a^2/r^2)(1 - h^2 - k^2)(c_F - k) + X_1/a + 2k}{(ks_F - hc_F)(1 - h^2 - k^2)^{\frac{1}{2}}} \tag{55}$$

We now complete the derivation of the partials of L with respect to the integrated state variables by generating $\partial L/\partial\lambda$. From $L = \omega + \Omega + \theta^* = \tan^{-1}(h/k) + \theta^*$, we have $\partial L/\partial\lambda = \partial\theta^*/\partial\lambda$. We also have $\partial r/\partial\lambda = (\partial r/\partial F)\,(\partial F/\partial\lambda) = a(ks_F - hc_F)\cdot\partial F/\partial\lambda = (a^2/r)(ks_F - hc_F)$ because $\partial F/\partial\lambda = a/r$. From $r = a(1 - e^2)/(1 + ec_{\theta^*})^2$, we have

$$\frac{\partial r}{\partial\lambda} = \frac{ae(1 - e^2)s_{\theta^*}(\partial\theta^*/\partial\lambda)}{(1 + ec_{\theta^*})^2} = \frac{a^2}{r}(ks_F - hc_F)$$

which yields

$$\frac{\partial\theta^*}{\partial\lambda} = \frac{a^2}{r^2}(1 - e^2)^{\frac{1}{2}}$$

and, therefore,

$$\frac{\partial L}{\partial\lambda} = \frac{a^2}{r^2}(1 - h^2 - k^2)^{\frac{1}{2}} \tag{56}$$

This also shows that $\partial L/\partial\lambda = (\partial L/\partial F)\cdot(\partial F/\partial\lambda)$. The partial $\partial r/\partial\lambda$ can be written in terms of L as

$$\frac{\partial r}{\partial\lambda} = \frac{a(ks_L - hc_L)}{(1 - h^2 - k^2)^{\frac{1}{2}}} \tag{57}$$

We have thus generated all of the partial derivatives of L, namely, $\partial L/\partial a$, $\partial L/\partial h$, $\partial L/\partial k$, and $\partial L/\partial\lambda$, given by Eqs. (48), (50), (54), and (56), respectively, as well as the partials of r, namely, $\partial r/\partial a$, $\partial r/\partial h$, $\partial r/\partial k$, and $\partial r/\partial\lambda$ in Eqs. (32), (39), (40), and (57), respectively. We can now generate all of the $\partial B/\partial z$ partials of interest using

$$\frac{\partial n}{\partial a} = -\frac{3n}{2a} = -\frac{3}{2}\mu^{\frac{1}{2}}a^{-\frac{5}{2}} \tag{58}$$

The nonzero elements of matrix B and their nonzero partials, with $G = (1 - h^2 - k^2)^{1/2}$ and $K = (1 + p^2 + q^2)$, are shown in the Appendix.

III. Numerical Example

Let us now duplicate an example of a minimum-time low-Earth-orbit-to-geostationary-Earth-orbit transfer that appeared in Ref. 7. The minimization of t_f or maximization of $-t_f$ requires the satisfaction of the transversality condition at the guessed final time $H_f = 1$ for the Hamiltonian used in Eq. (27). The orientation of the constant acceleration vector is optimized by selecting $\hat{u} = [\boldsymbol{\lambda}_z^T B(z, L)]^T/|\boldsymbol{\lambda}_z^T B(z, L)|$. Given a_0, e_0, i_0, Ω_0, and ω_0 or in an equivalent way $(a)_0$, $(h)_0$, $(k)_0$, $(p)_0$, and $(q)_0$, the corresponding multipliers $(\lambda_a)_0$, $(\lambda_h)_0$, $(\lambda_k)_0$, $(\lambda_p)_0$, and $(\lambda_q)_0$, as well as the transfer time t_f and the initial mean longitude $(\lambda)_0$, are guessed, and the dynamic equations (13–18) and the adjoint equations (28) are integrated forward with $(\lambda_\lambda)_0 = 0$, until t_f. The first five multipliers, $(\lambda)_0$, and t_f are then adjusted until a_f, h_f, k_f, p_f, q_f, $(\lambda_\lambda)_f = 0$, and $H_f = 1$ are satisfied for an optimal free-free transfer. By a free-free transfer, we mean a transfer where the first five elements defining the orbit size, shape, and orientation are fixed and the location or longitude of the spacecraft is free, on both the initial and final orbits. An unconstrained minimization algorithm is used to this effect and the following objective function minimized:

$$F' = \omega_1(a - a_f)^2 + \omega_2(h - h_f)^2 + \omega_3(k - k_f)^2 + \omega_4(p - p_f)^2 + \omega_5(q - q_f)^2 + \omega_6(\lambda_\lambda - 0)^2 + \omega_7(H - 1)^2$$

The ω_i weights had to be adjusted manually between consecutive runs to control the convergence process. The minimization subroutine uses a quasi-Newton algorithm and is based on a general descent method. No attempt was made to apply the multiple shooting method, which could, in theory, increase the robustness of the iterative convergence process. However, as was demonstrated in Ref. 7, the transfer can be solved first by using the averaging technique in a robust way, and the initial values of the Lagrange multipliers thus generated are used as a starting guess to solve the exact unaveraged transfer much more efficiently because these starting values are very close to the exact solution. Let $a_0 = 7000$ km, $e_0 = 0$, $i_0 = 28.5$ deg, $\Omega_0 = 0$ deg, and $\omega_0 = 0$ deg and let $a_f = 42{,}000$ km, $e_f = 10^{-3}$, $i_f = 1$ deg, $\Omega_f = 0$ deg, $\omega_f = 0$ deg, and $f_t = 9.8 \times 10^{-5}$ km/s^2. The solution is given by $(\lambda_a)_0 = 4.675229762$ s/km, $(\lambda_h)_0 = 5.413413947 \times 10^2$ s, $(\lambda_k)_0 = -9.202702084 \times 10^3$ s, $(\lambda_p)_0 = 1.778011878 \times 10^1$ s, $(\lambda_q)_0 = -2.258455855 \times 10^4$ s, $(\lambda)_0 = -2.274742851$ rad, which corresponds to $M_0 = -130.333164$ deg, and $t_f = 58{,}089.90058$ s. The final achieved parameters are $a_f = 42{,}000.003$ km, $e_f = 9.986352 \times 10^{-4}$, $i_f = 0.999809$ deg, $\Omega_f = 1.099680$ deg $\times 10^{-4}$, $\omega_f = 1.827518$ deg $\times 10^{-2}$, $(\lambda_\lambda)_f = -1.823951 \times 10^{-4}$ s/rad, and $H_f = 1.003704$ indicating convergence. Also, $\lambda_f = 19.655283$ rad, corresponding to $M_f = 46.146408$ deg. The initial values of the multipliers are essentially identical, as they should be because of their definition as influence functions, to the solution obtained in Ref. 7, thereby validating the mathematical derivations produced in this paper.

IV. Concluding Remarks

The mathematical derivations leading to the generation of the complete set of the dynamic and adjoint differential equations needed in trajectory optimization work are presented for a set of nonsingular orbit elements and for the resolution of the perturbation accelerations in the Euler–Hill rotating orbital frame. The element set adopted here is that originally used by Broucke and Cefola[1] for which the perturbation equations for the geopotential, drag, and third body effects were developed. These analytic methods derived from celestial mechanics can be put to effective use in constructing efficient optimization computer programs whether based on direct or indirect methods. Finally, an example of a free-free orbit transfer is duplicated with the present formulation in order to validate this simpler version, which also replaces the eccentric longitude by the true longitude as the accessory variable appearing in the right-hand side of all of the variational and adjoint equations.

Appendix: *B* Matrix and Its Partial Derivatives

$$B_{11} = 2n^{-1}G^{-1}(ks_L - hc_L)$$

$$\frac{\partial B_{11}}{\partial a} = -2n^{-2}\frac{\partial n}{\partial a}G^{-1}(ks_L - hc_L) \tag{A1}$$

$$\frac{\partial B_{11}}{\partial h} = 2n^{-1}hG^{-3}(ks_L - hc_L) + 2n^{-1}G^{-1} \times \left[-c_L + (hs_L + kc_L)\frac{\partial L}{\partial h}\right] \tag{A2}$$

$$\frac{\partial B_{11}}{\partial k} = 2n^{-1}kG^{-3}(ks_L - hc_L) + 2n^{-1}G^{-1} \times \left[s_L + (kc_L + hs_L)\frac{\partial L}{\partial k}\right] \tag{A3}$$

$$\frac{\partial B_{11}}{\partial \lambda} = 2n^{-1}G^{-1}(kc_L + hs_L)\frac{\partial L}{\partial \lambda} \tag{A4}$$

$$B_{12} = 2n^{-1}ar^{-1}G$$

$$\frac{\partial B_{12}}{\partial a} = -2n^{-2}ar^{-1}\frac{\partial n}{\partial a}G \tag{A5}$$

$$\frac{\partial B_{12}}{\partial h} = -2n^{-1}ar^{-2}\frac{\partial r}{\partial h}G - 2n^{-1}ar^{-1}hG^{-1} \tag{A6}$$

$$\frac{\partial B_{12}}{\partial k} = -2n^{-1}ar^{-2}\frac{\partial r}{\partial k}G - 2n^{-1}ar^{-1}kG^{-1} \tag{A7}$$

$$\frac{\partial B_{12}}{\partial \lambda} = -2n^{-1}ar^{-2}\frac{\partial r}{\partial \lambda}G \tag{A8}$$

$$B_{21} = -n^{-1}a^{-1}Gc_L$$

$$\frac{\partial B_{21}}{\partial a} = -\frac{1}{2}n^{-1}a^{-2}Gc_L \tag{A9}$$

$$\frac{\partial B_{21}}{\partial h} = n^{-1}a^{-1}hG^{-1}c_L + n^{-1}a^{-1}Gs_L\frac{\partial L}{\partial h} \tag{A10}$$

$$\frac{\partial B_{21}}{\partial k} = n^{-1}a^{-1}kG^{-1}c_L + n^{-1}a^{-1}Gs_L\frac{\partial L}{\partial k} \tag{A11}$$

$$\frac{\partial B_{21}}{\partial \lambda} = n^{-1}a^{-1}Gs_L\frac{\partial L}{\partial \lambda} \tag{A12}$$

$$B_{22} = rn^{-1}a^{-2}G^{-1}(h + s_L) + n^{-1}a^{-1}Gs_L$$

$$\frac{\partial B_{22}}{\partial a} = \frac{1}{2}n^{-1}a^{-3}rG^{-1}(h + s_L) + \frac{1}{2}n^{-1}a^{-2}Gs_L \tag{A13}$$

$$\frac{\partial B_{22}}{\partial h} = n^{-1}a^{-2}G^{-1}(h + s_L)\left[\frac{\partial r}{\partial h} + rhG^{-2}\right] + n^{-1}a^{-2}rG^{-1}\left(1 + c_L\frac{\partial L}{\partial h}\right) - n^{-1}a^{-1}hs_LG^{-1} + n^{-1}a^{-1}Gc_L\frac{\partial L}{\partial h} \tag{A14}$$

$$\frac{\partial B_{22}}{\partial k} = n^{-1}a^{-2}G^{-1}(h + s_L)\left[\frac{\partial r}{\partial k} + rkG^{-2}\right] + n^{-1}a^{-2}rG^{-1}c_L\frac{\partial L}{\partial k} - n^{-1}a^{-1}ks_LG^{-1} + n^{-1}a^{-1}Gc_L\frac{\partial L}{\partial k} \tag{A15}$$

$$\frac{\partial B_{22}}{\partial \lambda} = \frac{\partial r}{\partial \lambda}n^{-1}a^{-2}G^{-1}(h + s_L) + rn^{-1}a^{-2}G^{-1}c_L\frac{\partial L}{\partial \lambda} + n^{-1}a^{-1}Gc_L\frac{\partial L}{\partial \lambda} \tag{A16}$$

$$B_{23} = -n^{-1}a^{-2}rG^{-1}k(pc_L - qs_L)$$

$$\frac{\partial B_{23}}{\partial a} = -\frac{1}{2}n^{-1}a^{-3}rG^{-1}k(pc_L - qs_L) \tag{A17}$$

$$\frac{\partial B_{23}}{\partial h} = -n^{-1}a^{-2}G^{-1}k(pc_L - qs_L)\left[\frac{\partial r}{\partial h} + hrG^{-2}\right] + n^{-1}a^{-2}rkG^{-1}(ps_L + qc_L)\frac{\partial L}{\partial h} \tag{A18}$$

$$\frac{\partial B_{23}}{\partial k} = -n^{-1}a^{-2}G^{-1}k(pc_L - qs_L)\left[\frac{\partial r}{\partial k} + krG^{-2}\right] + n^{-1}a^{-2}rkG^{-1}(ps_L + qc_L)\frac{\partial L}{\partial k} - n^{-1}a^{-2}rG^{-1}(pc_L - qs_L) \tag{A19}$$

$$\frac{\partial B_{23}}{\partial p} = -n^{-1}a^{-2}rG^{-1}kc_L \tag{A20}$$

$$\frac{\partial B_{23}}{\partial q} = n^{-1}a^{-2}rG^{-1}ks_L \tag{A21}$$

$$\frac{\partial B_{23}}{\partial \lambda} = -n^{-1}a^{-2}\frac{\partial r}{\partial \lambda}G^{-1}k(pc_L - qs_L) + n^{-1}a^{-2}rG^{-1}k(ps_L + qc_L)\frac{\partial L}{\partial \lambda} \tag{A22}$$

$$B_{31} = n^{-1}a^{-1}Gs_L$$

$$\frac{\partial B_{31}}{\partial a} = \frac{1}{2}n^{-1}a^{-2}Gs_L \tag{A23}$$

$$\frac{\partial B_{31}}{\partial h} = -n^{-1}a^{-1}G^{-1}hs_L + n^{-1}a^{-1}Gc_L\frac{\partial L}{\partial h} \tag{A24}$$

$$\frac{\partial B_{31}}{\partial k} = -n^{-1}a^{-1}G^{-1}ks_L + n^{-1}a^{-1}Gc_L\frac{\partial L}{\partial k} \tag{A25}$$

$$\frac{\partial B_{31}}{\partial \lambda} = n^{-1}a^{-1}Gc_L\frac{\partial L}{\partial \lambda} \tag{A26}$$

$$B_{32} = n^{-1}a^{-2}rG^{-1}(k + c_L) + n^{-1}a^{-1}Gc_L$$

$$\frac{\partial B_{32}}{\partial a} = \frac{1}{2}n^{-1}a^{-3}rG^{-1}(k + c_L) + \frac{1}{2}n^{-1}a^{-2}Gc_L \tag{A27}$$

$$\frac{\partial B_{32}}{\partial h} = n^{-1}a^{-2}G^{-1}(k + c_L)\left[\frac{\partial r}{\partial h} + hrG^{-2}\right] - n^{-1}a^{-2}rG^{-1}s_L\frac{\partial L}{\partial h} - n^{-1}a^{-1}hG^{-1}c_L - n^{-1}a^{-1}Gs_L\frac{\partial L}{\partial h} \tag{A28}$$

$$\frac{\partial B_{32}}{\partial k} = n^{-1}a^{-2}G^{-1}(k + c_L)\left[\frac{\partial r}{\partial k} + krG^{-2}\right] - n^{-1}a^{-2}rG^{-1}s_L\frac{\partial L}{\partial k} - n^{-1}a^{-1}kG^{-1}c_L - n^{-1}a^{-1}Gs_L\frac{\partial L}{\partial k} + n^{-1}a^{-2}rG^{-1} \tag{A29}$$

$$\frac{\partial B_{32}}{\partial \lambda} = n^{-1}a^{-2}\frac{\partial r}{\partial \lambda}G^{-1}(k + c_L) - n^{-1}a^{-2}rG^{-1}s_L\frac{\partial L}{\partial \lambda} - n^{-1}a^{-1}Gs_L\frac{\partial L}{\partial \lambda} \tag{A30}$$

$$B_{33} = n^{-1}a^{-2}rG^{-1}h(pc_L - qs_L)$$

$$\frac{\partial B_{33}}{\partial a} = \frac{1}{2}n^{-1}a^{-3}rG^{-1}h(pc_L - qs_L) \tag{A31}$$

$$\frac{\partial B_{33}}{\partial h} = n^{-1}a^{-2}G^{-1}h(pc_L - qs_L)\left[\frac{\partial r}{\partial h} + hrG^{-2}\right] + n^{-1}a^{-2}rG^{-1}\left[(pc_L - qs_L) - h(ps_L + qc_L)\frac{\partial L}{\partial h}\right] \tag{A32}$$

$$\frac{\partial B_{33}}{\partial k} = n^{-1}a^{-2}G^{-1}h(pc_L - qs_L)\left[\frac{\partial r}{\partial k} + krG^{-2}\right] - n^{-1}a^{-2}rG^{-1}h(ps_L + qc_L)\frac{\partial L}{\partial k} \tag{A33}$$

$$\frac{\partial B_{33}}{\partial p} = n^{-1}a^{-2}rG^{-1}hc_L \tag{A34}$$

$$\frac{\partial B_{33}}{\partial q} = -n^{-1}a^{-2}rG^{-1}hs_L \tag{A35}$$

$$\frac{\partial B_{33}}{\partial \lambda} = n^{-1}a^{-2}\frac{\partial r}{\partial \lambda}G^{-1}h(pc_L - qs_L) - n^{-1}a^{-2}rG^{-1}h(ps_L + qc_L)\frac{\partial L}{\partial \lambda} \tag{A36}$$

$$B_{43} = 2^{-1}n^{-1}a^{-2}rG^{-1}Ks_L$$

$$\frac{\partial B_{43}}{\partial a} = \frac{1}{4}n^{-1}a^{-3}rG^{-1}Ks_L \tag{A37}$$

$$\frac{\partial B_{43}}{\partial h} = \frac{1}{2}n^{-1}a^{-2}G^{-1}Ks_L\left[\frac{\partial r}{\partial h} + hrG^{-2}\right] + \frac{1}{2}n^{-1}a^{-2}rG^{-1}Kc_L\frac{\partial L}{\partial h} \tag{A38}$$

$$\frac{\partial B_{43}}{\partial k} = \frac{1}{2}n^{-1}a^{-2}G^{-1}Ks_L\left[\frac{\partial r}{\partial k} + krG^{-2}\right] + \frac{1}{2}n^{-1}a^{-2}rG^{-1}Kc_L\frac{\partial L}{\partial k} \tag{A39}$$

$$\frac{\partial B_{43}}{\partial p} = n^{-1}a^{-2}rG^{-1}ps_L \tag{A40}$$

$$\frac{\partial B_{43}}{\partial q} = n^{-1}a^{-2}rG^{-1}qs_L \tag{A41}$$

$$\frac{\partial B_{43}}{\partial \lambda} = \frac{1}{2}n^{-1}a^{-2}\frac{\partial r}{\partial \lambda}G^{-1}Ks_L + \frac{1}{2}n^{-1}a^{-2}rG^{-1}Kc_L\frac{\partial L}{\partial \lambda} \tag{A42}$$

$$B_{53} = \frac{1}{2}n^{-1}a^{-2}rG^{-1}Kc_L$$

$$\frac{\partial B_{53}}{\partial a} = \frac{1}{4}n^{-1}a^{-3}rG^{-1}Kc_L \tag{A43}$$

$$\frac{\partial B_{53}}{\partial h} = \frac{1}{2}n^{-1}a^{-2}G^{-1}Kc_L\left[\frac{\partial r}{\partial h} + hrG^{-2}\right] - \frac{1}{2}n^{-1}a^{-2}rG^{-1}Ks_L\frac{\partial L}{\partial h} \tag{A44}$$

$$\frac{\partial B_{53}}{\partial k} = \frac{1}{2}n^{-1}a^{-2}G^{-1}Kc_L\left[\frac{\partial r}{\partial k} + krG^{-2}\right] - \frac{1}{2}n^{-1}a^{-2}rG^{-1}Ks_L\frac{\partial L}{\partial k} \tag{A45}$$

$$\frac{\partial B_{53}}{\partial p} = n^{-1}a^{-2}rG^{-1}pc_L \tag{A46}$$

$$\frac{\partial B_{53}}{\partial q} = n^{-1}a^{-2}rG^{-1}qc_L \tag{A47}$$

$$\frac{\partial B_{53}}{\partial \lambda} = \frac{1}{2}n^{-1}a^{-2}\frac{\partial r}{\partial \lambda}G^{-1}Kc_L - \frac{1}{2}n^{-1}a^{-2}rG^{-1}Ks_L\frac{\partial L}{\partial \lambda} \tag{A48}$$

$$B_{61} = -n^{-1}a^{-1}(1 - \beta)(hs_L + kc_L) - 2n^{-1}a^{-2}r$$

$$\frac{\partial B_{61}}{\partial a} = -\frac{1}{2}n^{-1}a^{-2}(1 - \beta)(hs_L + kc_L) - n^{-1}a^{-3}r \tag{A49}$$

$$\frac{\partial B_{61}}{\partial h} = n^{-1}a^{-1}\frac{h\beta^3}{1 - \beta}(hs_L + kc_L) - n^{-1}a^{-1}(1 - \beta) \times \left[s_L + (hc_L - ks_L)\frac{\partial L}{\partial h}\right] - 2n^{-1}a^{-2}\frac{\partial r}{\partial h} \tag{A50}$$

$$\frac{\partial B_{61}}{\partial k} = n^{-1}a^{-2}\frac{k\beta^3}{1 - \beta}(hs_L + kc_L) - 2n^{-1}a^{-2}\frac{\partial r}{\partial k} - n^{-1}a^{-1}(1 - \beta)\left[c_L + (hc_L - ks_L)\frac{\partial L}{\partial k}\right] \tag{A51}$$

$$\frac{\partial B_{61}}{\partial \lambda} = -n^{-1}a^{-1}(1-\beta)(hc_L - ks_L)\frac{\partial L}{\partial \lambda} - 2n^{-1}a^{-2}\frac{\partial r}{\partial \lambda} \quad \text{(A52)}$$

$$B_{62} = -n^{-1}a^{-1}(1-\beta)(hc_L - ks_L)[1 + ra^{-1}G^{-2}]$$

$$\frac{\partial B_{62}}{\partial a} = -\frac{1}{2}n^{-1}a^{-2}(1-\beta)(hc_L - ks_L)[1 + ra^{-1}G^{-2}] \quad \text{(A53)}$$

$$\frac{\partial B_{62}}{\partial h} = n^{-1}a^{-1}\frac{h\beta^3}{1-\beta}(hc_L - ks_L)[1 + ra^{-1}G^{-2}]$$

$$- n^{-1}a^{-1}(1-\beta)\left[c_L - (hs_L + kc_L)\frac{\partial L}{\partial h}\right][1 + ra^{-1}G^{-2}]$$

$$- n^{-1}a^{-2}(1-\beta)(hc_L - ks_L)G^{-2}\left[\frac{\partial r}{\partial h} + 2hrG^{-2}\right] \quad \text{(A54)}$$

$$\frac{\partial B_{62}}{\partial k} = n^{-1}a^{-1}\frac{k\beta^3}{1-\beta}(hc_L - ks_L)[1 + ra^{-1}G^{-2}]$$

$$+ n^{-1}a^{-1}(1-\beta)\left[s_L + (hs_L + kc_L)\frac{\partial L}{\partial k}\right][1 + ra^{-1}G^{-2}]$$

$$- n^{-1}a^{-2}(1-\beta)(hc_L - ks_L)G^{-2}\left[\frac{\partial r}{\partial k} + 2krG^{-2}\right] \quad \text{(A55)}$$

$$\frac{\partial B_{62}}{\partial \lambda} = n^{-1}a^{-1}(1-\beta)(hs_L + kc_L)\frac{\partial L}{\partial \lambda}[1 + ra^{-1}G^{-2}]$$

$$- n^{-1}a^{-2}(1-\beta)(hc_L - ks_L)\frac{\partial r}{\partial \lambda}G^{-2} \quad \text{(A56)}$$

$$B_{63} = -n^{-1}a^{-2}rG^{-1}(pc_L - qs_L)$$

$$\frac{\partial B_{63}}{\partial a} = -\frac{1}{2}n^{-1}a^{-3}rG^{-1}(pc_L - qs_L) \quad \text{(A57)}$$

$$\frac{\partial B_{63}}{\partial h} = -n^{-1}a^{-2}G^{-1}(pc_L - qs_L)\left[\frac{\partial r}{\partial h} + hrG^{-2}\right]$$

$$+ n^{-1}a^{-2}rG^{-1}(ps_L + qc_L)\frac{\partial L}{\partial h} \quad \text{(A58)}$$

$$\frac{\partial B_{63}}{\partial k} = -n^{-1}a^{-2}G^{-1}(pc_L - qs_L)\left[\frac{\partial r}{\partial k} + krG^{-2}\right]$$

$$+ n^{-1}a^{-2}rG^{-1}(ps_L + qc_L)\frac{\partial L}{\partial k} \quad \text{(A59)}$$

$$\frac{\partial B_{63}}{\partial p} = -n^{-1}a^{-2}rG^{-1}c_L \quad \text{(A60)}$$

$$\frac{\partial B_{63}}{\partial q} = n^{-1}a^{-2}rG^{-1}s_L \quad \text{(A61)}$$

$$\frac{\partial B_{63}}{\partial \lambda} = -n^{-1}a^{-2}\frac{\partial r}{\partial \lambda}G^{-1}(pc_L - qs_L)$$

$$+ n^{-1}a^{-2}rG^{-1}(ps_L + qc_L)\frac{\partial L}{\partial \lambda} \quad \text{(A62)}$$

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